Korteweg–de Vries–Burgers equation for surface waves in nonideal conducting liquids

A. González[1,2] and A. Castellanos[1]

[1] *Departamento de Electrónica y Electromagnetismo, Universidad de Sevilla, Avenida Reina Mercedes, s/n 41012 Sevilla, Spain*

[2] *Departamento de Física Aplicada, Universidad de Sevilla, Avenida Reina Mercedes, s/n 41012 Sevilla, Spain*

(Received 8 June 1993; revised manuscript received 29 November 1993)

We examine the propagation of weakly nonlinear waves on the surface of a shallow viscous conducting liquid layer stressed by a normal electric field. A Korteweg–de Vries–Burgers equation is obtained using multiple-scale analysis and a systematic treatment. We discuss the possibility of sustaining solitons of permanent amplitude with the aid of electric fields to compensate viscous losses.

PACS number(s): 47.65.+a, 47.35.+i

I. INTRODUCTION

The existence of solitary waves in dissipative media has recently been investigated, e.g. for thin layers where viscous friction cannot be disregarded. It is known that these nonlinear waves can be excited by mechanical forces, e.g., gravitational forces for inclined liquid layers [1,2], or Marangoni forces in horizontal layers [3]. Recently, Weidman, Linde, and Velarde [4] have reviewed experiments performed long ago by Linde and collaborators and have proven the existence of solitary waves in a shallow liquid layer, driven by Marangoni forces. The properties of these waves resemble those of solitons in inviscid liquids.

Contrary to the case of solitons, these dissipative waves are not dependent on initial conditions, but rather on the balance of injection of energy, dissipation, nonlinearity, and dispersion. There is no theory to *quantitatively* predict the nature of these waves. Nevertheless, some features have been predicted by numerical and theoretical analysis based on simplified models [1–3].

The electric field could be a suitable physical agent to generate solitary waves in horizontal layers. Electrical energy may be pumped into liquids with finite conductivity. However, as shown by Melcher [5], a finite conductivity unavoidably implies the consideration of viscosity to compensate tangential electrical stresses on the free surface.

Recently, Easwaran [6] considered the behavior of weakly nonlinear long waves on a shallow perfectly conducting liquid layer. He deduced a Korteweg–deVries (KdV) equation for the surface deviation, with coefficients modified by the electric field.

Here we consider two effects: viscosity in the bulk and finite electrical conductivity. A systematic approach based on the Fredholm alternative is used to derive the equation satisfied by the deflection of the interface. The aim is to show the possibility of sustaining solitary waves, compensating the viscous losses by injecting electrical energy into the system. When viscosity effects are of the same order as nonlinearity and dispersion, we obtain a Korteweg–de Vries–Burgers (KdVB) equation. This equation admits kink type solutions of permanent form. However, solitons of finite length would be either damped or amplified, depending on the ratio of electrical energy input to viscous losses, and consequently show the existence of a *threshold* value of the electric field, similar to the case of Marangoni convection [7]. Above this value, the theory presented here cannot follow the further evolution of the unstable waves. Higher order terms should then be considered to describe this case. The existence of this threshold might be indicative of the existence of solitary waves excited by electric fields in viscous liquids.

II. SYSTEM DESCRIPTION

Let us consider a liquid layer of thickness d resting on a grounded conducting electrode at $z = -d$. Another electrode, with fixed potential V_0, is placed at a distance $z = h$ above the surface. The liquid is assumed to be incompressible and to have density ρ, kinematic viscosity ν, large, but finite, electrical conductivity σ, and electrical permittivity ε. The system is subjected to a vertical gravitational acceleration g.

The system extends infinitely in the horizontal directions but only disturbances depending on the x coordinate are considered. The surface elevation can then be described by a function $\eta(x,t)$. The amplitude of these disturbances is of order a, a length usually much smaller than the depth.

Let u and w denote the horizontal and vertical velocity components; p the excess over the hydrostatic pressure; and ϕ and ψ the excess over the static electric potential in the liquid and in the air, respectively.

The evolution of these magnitudes is governed by a set of equations that we present in nondimensional form.

The units used to make the variables dimensionless are as follows. The scales for the mechanical magnitudes are well known, and can be obtained from dimensional linear analysis,

$$x = \lambda, \quad z = d, \quad \eta = a, \quad t = T, \tag{1a}$$

$$u = \frac{a\lambda}{Td}, \quad w = \frac{a}{T}, \quad p = \rho g a, \tag{1b}$$

where T is a typical mechanical time. The simplest choice is given by the ratio between the wavelength of the disturbances and their linear velocity, i.e., the period of the linear waves $T = \lambda/(gd)^{1/2}$.

Appropriate scales for the potential ϕ and ψ are similarly derived from the linearized equations for the continuity of the potential and the charge conservation equation

$$\begin{array}{ccccc} \phi & = & \psi & + & V_0 \dfrac{\eta}{h}, \\ [\phi_0] & & [\psi_0] & & \left[\dfrac{a}{h} V_0\right] \end{array} \tag{2}$$

$$\begin{array}{ccccccccc} -\varepsilon_0 \dfrac{\partial^2 \psi}{\partial z \partial t} & + & \varepsilon \dfrac{\partial^2 \phi}{\partial z \partial t} & + & \sigma \dfrac{\partial \phi}{\partial z} & - & \dfrac{\partial}{\partial x}\left(\varepsilon_0 \dfrac{V_0}{h} u\right) & = 0. \\ \left[\dfrac{\varepsilon_0 \psi_0}{Th}\right] & & \left[\dfrac{\varepsilon \phi_0}{Td}\right] & & \left[\dfrac{\sigma \phi_0}{d}\right] & & \left[\dfrac{\varepsilon_0 a V_0}{Tdh}\right]. & \end{array} \tag{3}$$

The order of magnitude of each quantity is indicated in brackets. Multiplying the second one by d/σ, and taking into account that the conductivity is large enough to make $\varepsilon_0/\sigma T \ll 1$ (the charge relaxation time is assumed to be much smaller than the mechanical one), we arrive at the relation

$$\phi_0 \sim \frac{\varepsilon_0}{\sigma T} \frac{d}{h} \psi_0 \sim \frac{\varepsilon_0}{\sigma T} \psi_0 . \tag{4}$$

The following scales are finally obtained from the continuity of the potential:

$$\psi_0 = \frac{a}{h} V_0 , \qquad \phi_0 = \frac{\varepsilon_0}{\sigma T} \frac{a}{h} V_0 . \tag{5}$$

These orders of magnitudes imply that, at least to the first order, the liquid appears, from the air, to be a perfect conductor. They also imply that the field inside the liquid is generated by changes in the surface charge.

The system of dimensionless equations (the parameters will be explained later) is given by

(1) Continuity and Navier-Stokes equations:

$$u_x + w_z = 0 , \tag{6a}$$

$$u_t + \alpha(u u_x + w u_z) = -p_x + V(u_{zz} + \delta^2 u_{xx}) , \tag{6b}$$

$$\delta^2[w_t + \alpha(u w_x + w w_z)] = -p_z + \delta^2 V(w_{zz} + \delta^2 w_{xx}) ; \tag{6c}$$

(2) Laplace equations for the electric potentials:

$$\phi_{zz} + \delta^2 \phi_{xx} = 0 , \tag{6d}$$

$$\psi_{zz} + \delta^2 \psi_{xx} = 0 . \tag{7}$$

Equations (6) apply to the liquid layer ($-1 < z < \alpha\eta$) and (7) to the air layer ($\alpha\eta < z < D$).

We must add the following boundary conditions to this set of equations.

(1) At the lower plate ($z = -1$) we have

$$w = 0 , \quad u_z = 0 , \quad \phi = 0 . \tag{8}$$

Note that the second condition implies that the sliding resistance between two portions of the liquid is much greater than that between the liquid and the lower plate. With this assumption, the boundary condition is a "stress-free" condition and the viscous effects mainly take place in the bulk. The more realistic condition $u = 0$ ("no-slip" condition) makes the problem much more difficult. Because of this, the stress-free condition has often been used [3].

(2) The sole condition at the upper plate ($z = D$) is

$$\psi = 0 . \tag{9}$$

At the liquid surface ($z = \alpha\eta$) we have five conditions.

(1) Kinematical condition

$$w = \eta_t + \alpha u \eta_x . \tag{10a}$$

(2) Continuity of the potential

$$R_E \phi = \psi + \eta . \tag{10b}$$

(3) Charge conservation equation

$$\left(-\frac{1}{N\alpha} + q\right)_t + \left[\left(-\frac{1}{N} + \alpha q\right)\left(\frac{u + \alpha\delta^2 \eta_x w}{N}\right)\right]_s + 2\alpha\delta^2 \left(\frac{1}{N} - \alpha q\right) \frac{\eta_{xx}\eta_t}{N^3} + \phi_n = 0 , \tag{10c}$$

where q is the excess over the static surface charge

$$q = -(\psi - \varepsilon_r R_E \phi)_n \tag{10d}$$

and $\partial/\partial s$ and $\partial/\partial n$ (which appear as subscripts) denote the tangential and the normal derivative, respectively.

$$\frac{\partial}{\partial s} = \frac{1}{N}\left(\frac{\partial}{\partial x} + \alpha\eta_x \frac{\partial}{\partial z}\right), \quad \frac{\partial}{\partial n} = \frac{1}{N}\left(\frac{\partial}{\partial z} - \alpha\delta^2 \eta_x \frac{\partial}{\partial x}\right), \tag{10e}$$

with

$$N = \left(1 + \alpha^2 \delta^2 \eta_x^2\right)^{1/2} . \tag{10f}$$

Note that the term describing the divergence of the surface currents has two parts: one corresponding to the divergence of the currents, as they are seen from the surface itself, and another associated with the change in the surface curvature.

(4) Continuity of the tangential stresses:

$$V[(u_n - \alpha\delta^2 \eta_x u_s) + \delta^2(w_s + \alpha\eta_x w_n)] + DWR_E \left(\frac{1}{N} - \alpha q\right) \phi_s = 0 . \tag{10g}$$

(5) Jump in the normal stresses:

$$WD\left[\frac{\alpha}{2}\left(\frac{1}{\alpha N}+\psi_n\right)^2-\frac{\varepsilon_r R_E^2\alpha}{2}\phi_n^2-(1-\varepsilon_r)\frac{\alpha\delta^2 R_E^2}{2}\phi_s^2\right]$$
$$+2V(w_n-\alpha\eta_x u_n)+p-\eta-\frac{WD}{2\alpha}=-\frac{2}{B}\frac{\eta_{xx}}{N^3}\,. \tag{10h}$$

The dimensionless parameters are the following:
(1) relative depth $\delta = d/\lambda$,
(2) relative thickness $D = h/d$,
(3) relative amplitude $\alpha = a/d$,
(4) viscous parameter $V = \nu T/d^2$,
(5) electric parameter $W = \varepsilon_0 V_0^2/\rho g h^3$,
(6) electric Reynolds number $R_E = \varepsilon_0/\sigma T$,
(7) relative permittivity $\varepsilon_r = \varepsilon/\varepsilon_0$,
(8) Bond number $B = \rho g \lambda^2/\gamma$.
The number of parameters is so large that many different analyses can be performed. In this paper, we restrict ourselves to a particular balance that we find appropriate to the case of viscous liquids subjected to achievable electric fields. A more general analysis, based on a multiple-parameter expansion, is left for further research.

The first three parameters are purely geometrical. The relative thickness will be a number of order unity. However, the relative depth will be assumed to be much smaller than one (shallow layer limit). The relative amplitude will also be much smaller than the unity, since the behavior is weakly nonlinear. In fact, we restrict ourselves to the case where $\alpha \sim \delta^2$. This is the limit for which nonlinearity and dispersion are of the same order and the KdV equation appears. We introduce the Ursell number $U = \alpha/\delta^2$ as an auxiliary parameter.

The viscous parameter V measures the ratio between viscous and inertial terms. We will assume that it is of order one. Since our boundary conditions do not induce vorticity in the liquid, the basic viscous solution will be the same as the inviscid one. Therefore, viscous effects will be of second order, despite the fact that $V \sim 1$.

The electric parameter measures the ratio between the variation caused by the disturbances in the electrical energy density [of order $\varepsilon_0 E_{st} E' \sim \varepsilon_0 (V_0/h)(aV_0/h^2)$] and the gravitational energy density (of order $\rho g a$). It is a number of order unity.

The electric Reynolds number R_E compares the charge relaxation time to the mechanical time and is much smaller than one, as previously stated. Since we want the pumping of electrical energy to be of the same order as the viscous losses, we assume that $R_E \sim \delta^2$.

Finally, as we deal with long waves, the Bond number B will be assumed to be of order δ^{-2}.

Summarizing, our assumptions regarding the orders of magnitude are

$$D \sim W \sim V \sim \varepsilon_r \sim 1\,,$$
$$\alpha \sim R_E \sim B^{-1} \sim \delta^2 \ll 1\,. \tag{11}$$

III. TREATMENT OF THE PROBLEM: THE LINEAR REGIME

We will proceed by repeated steps. First, we expand all the magnitudes and equations in powers of δ^2. We also expand the boundary conditions, evaluating them at $z = 0$ through a Taylor series. The expansions of the magnitudes have the form

$$A = \sum_i A^{(i)} \delta^{2i}\,. \tag{12}$$

We use the multiple-scale method to avoid possible secularities, following Nayfeh [8]. Thus we introduce a series of time scales $t_0, t_1, t_2, \ldots$ so that

$$A_t = A_{t_0} + \delta^2 A_{t_1} + \delta^4 A_{t_2} + \cdots\,. \tag{13}$$

We obtain a hierarchy of equations that gives the succesive corrections in the magnitudes. The derivatives with respect to the slow time scales are obtained from the solvability conditions that the variables and boundary conditions must satisfy. The Fredholm alternative is the method used to impose the solvability conditions.

The equations at the lower order are the well-known linear equations, which can be written as $\mathcal{L}\mathbf{u}^{(0)} = 0$, where $\mathbf{u}^{(0)} = (u^{(0)}, w^{(0)}, p^{(0)}, \phi^{(0)}, \psi^{(0)})$ and $\mathcal{L}$ is a piecewise-defined linear operator, given by

$$\mathcal{L} = \left(\begin{array}{cccc|c} \partial_x & \partial_z & 0 & 0 & 0 \\ \partial_{t_0} - V\partial_{zz} & 0 & \partial_x & 0 & 0 \\ 0 & 0 & \partial_z & 0 & 0 \\ 0 & 0 & 0 & \partial_{zz} & 0 \\ \hline 0 & 0 & 0 & 0 & \partial_{zz} \end{array}\right) \tag{14}$$

where the upper block and the corresponding four components of $\mathbf{u}^{(0)}$ are defined in the region $-1 < z < 0$ and the lower block in the region $0 < z < D$. More generally, each hierarchy level has the form $\mathcal{L}\mathbf{u}^{(n)} = \mathbf{b}^{(n)}$, where $\mathbf{b}^{(n)}$ a function of the lower orders.

The boundary conditions to first order are given by

$$w^{(0)} = 0, \quad u_z^{(0)} = 0, \quad \phi^{(0)} = 0, \quad (z = -1), \tag{15}$$

$$\psi^{(0)} = 0\,, \quad (z = D), \tag{16}$$

$$w^{(0)} = \eta_{t_0}\,, \quad \psi^{(0)} = -\eta\,, \quad (z = 0), \tag{17}$$

$$\phi_z^{(0)} - u_x^{(0)} - \psi_{zt}^{(0)} = 0\,, \quad (z = 0), \tag{18}$$

$$u_z^{(0)} = 0\,, \quad WD\psi_z^{(0)} + p^{(0)} = \eta, \quad (z = 0)\,. \tag{19}$$

Note that the interface elongation η is not expanded in powers of δ^2. Thus, η must satisfy a series of conditions that determines the evolution of the surface.

To determine the solvability condition we use the expression

$$\langle \mathbf{u}^\dagger, \mathcal{L}\mathbf{u}^{(n)} \rangle - \langle \mathcal{L}^\dagger \mathbf{u}^\dagger, \mathbf{u}^{(n)} \rangle = J(\mathbf{u}^\dagger, \mathbf{u}^{(n)})\,, \tag{20}$$

where $\mathcal{L}^\dagger$ denotes the adjoint operator of $\mathcal{L}$ under the scalar product

$$\langle \mathbf{v}^\dagger, \mathbf{u} \rangle = \lim_{\substack{L\to\infty \\ T\to\infty}} \int_{-L}^{L} dx \int_{-T}^{T} dt_0 \left[\int_{-1}^{0} dz \sum_{i=1}^{4} v_i^\dagger u_i^\dagger + \int_0^D dz\, v_5^\dagger u_5 \right] \tag{21}$$

and $\mathbf{u}^\dagger = (p^\dagger, u^\dagger, w^\dagger, \phi^\dagger, \psi^\dagger)$ is a vector of the dual space. The equations for the adjoint problem are obtained integrating by parts $\langle \mathbf{u}^\dagger, \mathcal{L}\mathbf{u}^{(n)} \rangle$. In addition, $J(\mathbf{u}^\dagger, \mathbf{u}^{(n)})$ in (20) denotes an integral of the exact differential of the left-hand side. It contains only terms evaluated at the boundaries. It can be used to choose appropriate boundary conditions for the adjoint problem. These must be those that make J vanish, assuming that the conditions of the direct problem are homogeneous.

The adjoint problem is easily solved and the solution is proportional to an arbitrary function Λ of x and t_0.

Substituting and integrating by parts in (20) results in an expression multiplied by Λ. Since this function is arbitrary, its coefficient must vanish, therefore leading to the solvability condition.

For $n = 0$, the solvability condition reduces to

$$\eta_{xx} - \frac{1}{1-W}\eta_{t_0 t_0} = 0 , \qquad (22)$$

therefore showing that a liquid in the shallow layer limit subjected to a normal electric field obeys a nondispersive wave equation. The main effect of the electric field is to change the phase speed, to $c = (1 - W)^{1/2}$ instead of $c = 1$, as would be found in the non-electrical problem. For $W = 1$, the speed vanishes and the system becomes unstable. The critical value of the electric field is identical to that previously determined by Taylor and McEwan [9] in the limit of long waves. We assume we are below this limit in the following.

Assuming that Eq. (22) is satisfied, we can obtain the explicit solution for the first-order direct problem. This is

$$u^{(0)} = \Omega_x, \quad w^{(0)} = -\Omega_{xx}(z+1), \quad p^{(0)} = -\Omega_{t_0} , \qquad (23)$$

$$\phi^{(0)} = \Omega_{xx}\frac{D-1}{D}(z+1), \quad \psi^{(0)} = \Omega_{t_0}\frac{1}{1-W}(z-D) , \qquad (24)$$

where Ω is a function related to η through $\eta = \Omega_{t_0}/(1 - W)$.

It is of interest to note that this solution is the same as in the inviscid case. This is a convenient consequence of the stress-free condition at the lower electrode.

Since the zero-order solvability condition is a linear nondispersive wave equation, it results that the surface elongation can be written as

$$\eta(x, t_0, t_1, \ldots) = \eta^+(x - ct_0, t_1, \ldots) + \eta^-(x + ct_0, t_1, \ldots), \qquad (25)$$

and the following relations hold:

$$\Omega_x = c(\eta^- - \eta^+), \quad \Omega_t = c^2(\eta^+ + \eta^-) . \qquad (26)$$

IV. THE WEAKLY NONLINEAR REGIME

Expanding the complete system up to the first power of δ^2, we obtain a new system of equations of the form $\mathcal{L}\mathbf{u}^{(1)} = \mathbf{b}^{(1)}$, where $\mathbf{b}^{(1)}$ is a vector dependent only on the known functions of order zero. This system need not be solved. Applying the Fredholm alternative is sufficient to obtain the equation for the evolution on the slow scale for the surface elevation.

We obtain the solvability condition making use of (20) for $n = 1$. This condition can be integrated once. The resulting equation is

$$\begin{aligned} 2\Omega_{xt_1} &+ 2U(\Omega_x\Omega_{xx}) \\ &+ \frac{U}{(1-W)}\left(1 - \frac{3W}{(1-W)D}\right)(\Omega_{t_0}\Omega_{xt_0}) \\ &- \left(4V + \frac{R_E W(D^2-1)}{D\delta^2}\right)\Omega_{xxx} \\ &- \left[\frac{1}{3}\left(1 + \frac{WD^2}{1-W}\right) + \frac{2}{B\delta^2(1-W)}\right]\Omega_{xxxt_0} = 0 . \end{aligned} \qquad (27)$$

This is a Boussinesq-type equation with a dissipative term. It contains terms associated to the nonlinearity (second and third), dissipation (fourth), and dispersion (fifth). In terms of the functions η^+ and η^-, this equation admits the form

$$\begin{aligned} &\left(\eta^+_{t_1} + A_1\eta^+\eta^+_\xi - A_2\eta^+_{\xi\xi} + A_3\eta^+_{\xi\xi\xi}\right) \\ &\quad - \left(\eta^-_{t_1} - A_1\eta^-\eta^-_\zeta - A_2\eta^-_{\zeta\zeta} - A_3\eta^+_{\zeta\zeta\zeta}\right) \\ &\quad - A_4\left(\eta^+\eta^-_\zeta + \eta^+_\xi\eta^-\right) = 0 , \end{aligned} \qquad (28)$$

where $\xi = x - ct_0$ and $\zeta = x + ct_0$ are the phases of the right- and left-moving linear waves, respectively.

The coefficients are given by

$$A_1 = \frac{3Uc}{2}\left(1 - \frac{W}{(1-W)D}\right) , \qquad (29a)$$

$$A_2 = 2V + \frac{R_E W(D^2-1)}{2\delta^2 D} , \qquad (29b)$$

$$A_3 = \frac{c}{6}\left(1 + \frac{WD^2}{1-W}\right) + \frac{1}{B\delta^2 c} , \qquad (29c)$$

$$A_4 = \frac{Uc}{2}\left(1 + \frac{3W}{(1-W)D}\right) . \qquad (29d)$$

The solvability condition, (28), can be considered as two coupled KdVB equations. Nevertheless, Byatt-Smith [10] has pointed out that a Boussinesq equation of the type (27) does not correctly describe the head-on collision of solitons, while the original system does make it.

If one of the traveling waves is absent, the equation reduces to a simple KdVB one. We assume this situation in the subsequent discussion

V. DISCUSSION OF THE RESULTS

The KdVB equation contains terms associated with the nonlinearity, dispersion, and dissipation. It is well known that in the absence of dissipation, the nonlinearity can compensate the dispersion and a solution is possible in the form of a "sech^2" soliton preserving its form. This solution no longer applies when there is dissipation because of the energy loses.

Several papers have been published regarding the search of exact solutions for this equation. In particular, Jeffrey and Xu [11] found two exact kinklike solutions. They have also shown that there are no more solutions of this type. Kudryashov [12] also found a solution which is the first of Jeffrey's.

These solutions are rather special because of their infinite length, which implies that an infinite amount of energy is fed into the system through the boundaries.

If we are interested in sustaining finite length waves, we must make the coefficient A_2 vanish. This coefficient reflects the balance between the viscous losses and the electrical energy input. This flux can have any sign, depending on the factor $(D^2 - 1)$. This factor comes from the charge conservation equation, since the electric field inside the liquid is caused by the charge variations at the surface.

For $D = 1$, the electrical energy injected into the fluid is zero. For this particular value of D, transport of surface charge by convection exactly compensates the temporal variation of the surface charge induced by the electrodes, therefore making the inner electric field zero.

To explain this phenomenon it is necessary to take into account that the changes in the surface charge are dominated by the variation of the electric field in the air, which is inversely proportional to the width of the air gap, whereas the transport of charge depends on the gradient of the horizontal velocity on the surface, which is inversely proportional to the depth of the liquid. Both factors have opposite signs. Therefore, when both thicknesses are equal, the two effects cancel one another and no electric field is generated inside the liquid.

If $d \neq h$, an electric field appears in the liquid, given approximately by

$$E_z \simeq \frac{\varepsilon_0 V_0}{\sigma h} \frac{\partial \eta}{\partial t} \left(\frac{1}{h} - \frac{1}{d} \right) . \tag{30}$$

From this component, we can estimate the value of the tangential field and from this, the value of the electric input of energy, given approximately by

$$P \simeq -\frac{\varepsilon_0^2 V_0^2}{\sigma h^2 d} \left(\frac{1}{h} - \frac{1}{d} \right) \lim_{L \to \infty} \int_{-L}^{L} dx \left(\frac{\partial u}{\partial x} \right)^2 . \tag{31}$$

Therefore, it can have any sign, depending on the ratio of d to h.

For $D > 1$ we see that mechanical energy is extracted and converted to electrical energy, therefore aiding the damping due to viscous losses. The electrical energy opposes this damping only for $D < 1$. For each value of $D < 1$ there is a critical value for the electrical parameter W, at which an exact compensation of viscosity takes place, given by

$$W = \frac{4DV\delta^2}{R_E(1 - D^2)} . \tag{32}$$

For this value, A_2 vanishes and the equation reduces to two coupled KdV equations (or to a single one). This equation is also obtained neglecting viscous and finite conductivity effects and it is similar to that obtained by Easwaran [6].

The waves are amplified if the field makes W greater than this value. The amplitude grows and the liquid is accelerated, making the viscous damping increase. Finally, a new equilibrium between forces might be reached. To describe this situation, new terms of the expansion must be included, since our analysis is only for weakly nonlinear waves. In any case, we believe that solitary waves may be sustained in a viscous liquid with an electric field.

An interesting feature of this KdV, already discussed by Easwaran, is the possibility of making the nonlinear term coefficient A_1 vanish. The value of the electric parameter at which this occurs is

$$W = \frac{D}{D + 1} . \tag{33}$$

This value is always less than the linear instability criterion. For this value only the dispersive term remains and solitary waves cannot exist. When the critical value is surpassed the sign of A_1 changes. This implies that for a given value of the soliton velocity, the amplitude is positive below and negative above (or vice versa). If the change is made slowly one can expect the soliton to accelerate (or decelerate) to reach the speed compatible with its amplitude.

We can estimate the values for the different magnitudes in a possible laboratory situation. Let us consider a layer of water of thickness 2 cm, under a layer of air of thickness 0.5 cm. For a perturbation of wavelength 10 cm, the linear instability requires a field of about 7.5×10^5 V/m. The critical value at which the nonlinear effects vanish lies in the range of 3.4×10^5 V/m and it is lower than the linear stability value. If the water is very pure, the electric field required to compensate the viscous losses in the bulk is even lower. Its value is close to 2×10^5 V/m. This means that the viscous damping can be compensated. "Antidissipative" effects could then be produced.

These values are not very far from the value of the breakdown field in wide air gaps. This problem could be avoided by choosing other liquids and gases with different properties.

ACKNOWLEDGMENTS

The authors are grateful to Professor Velarde for pointing out to us the existence of solitary waves in dissipative media excited by Marangoni forces. We have also had fruitful discussion with him about the role of the electric field. This work has been supported by the Spanish Dirección General Interministerial de Ciencia y Tecnología (DGICYT) under Contract No. PB90-0905.

[1] A. Pumir, P. Manneville, and Y. Pomeau, J. Fluid Mech. **135**, 27 (1983).
[2] H. Chang, Chem. Eng. Sci. **42**, 515 (1987).
[3] A. N. Garazo and M. G. Velarde, Phys. Fluids A **3**, 2295 (1991).
[4] P. Weidman, H. Linde, and M. G. Velarde, Phys. Fluids A **4**, 921 (1992).
[5] J. R. Melcher, *Field Coupled Surface Waves* (MIT Press, Cambridge, MA, 1963).
[6] C. V. Easwaran, Phys. Fluids **31**, 3442 (1988).
[7] X. L. Chu and M. G. Velarde, Phys. Rev. A **43**, 1094 (1991).
[8] A. H. Nayfeh, *Introduction to Perturbation Techniques* (Wiley Interscience, New York, 1981).
[9] G. I. Taylor and A. D. McEwan, J. Fluid Mech. **22**, 1 (1965).
[10] J. G. B. Byatt-Smith, J. Fluid Mech. **49**, 625 (1971).
[11] A. Jeffrey and S. Xu, Wave Motion **11**, 559 (1989).
[12] N. A. Kudryashov, Prikl. Matem. Mekhan. **52**, 465 (1988).